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Simultaneously estimating the initial and boundary conditions in a two-dimensional hollow cylinder

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Abstract—This study is intended to provide a different perspective for solving two-dimensional inverse heat conduction problems. At the beginning of the study, finite-difference methods are employed to discretize the problem domain and then a linear inverse model is constructed to identify the initial and boundary conditions. The present approach is to rearrange the matrix forms of differential governing equation and estimate coefficients of unknown condition. Then, the linear least-squares method is adopted to find the solution. The results show that if the measurement errors are considered, more measuring points are needed in order to increase the congruence of the estimated results to exact solutions. In this paper, temperature–time variations are measured at internal and outside points. A little effect of the measurement time interval on the estimates are shown with the method proposed. © 1997 Published by Elsevier Science Ltd.

INTRODUCTION

In recent years the analysis of inverse heat conduction problems (IHCP) has numerous applications in various branches of science and engineering, such as the prediction of the inner wall temperature of a reactor, the determination of the heat transfer coefficient, the outer surface conditions in the re-entry of a space vehicle, and the temperature or heat flux at the tool work interface of machine cutting. In most cases, the IHCP have basically dealt with one-dimensional (1-D) geometry. The difficulties of multi-dimensional IHCP are more pronounced, and little research is available, even for two-dimensional (2-D) cases.

Various methods have been employed to handle the IHCP in 1-D domains, such as graphical [1], polynomial [2–4], Laplace transform [5], finite difference and finite element [6–12], exact methods [13–14], and dynamic programming [15]. In contrast, in 2-D IHCP, the first analytical solution was introduced by Imber [16]. Subsequently, most of the research related to the numerical treatment of 2-D IHCP is based on different manners of combining finite-difference or finite-elements realizations with the future temperature method of Beck [9]. The applications of these ideas are presented in [17, 18]. More recently, a direct sensitivity coefficient method was presented by Tseng *et al.* [19].

In this study, a methodology is presented to solve the inverse problems. This method rearranges the matrix forms of direct problems in order to represent the unknown conditions explicitly. The inverse model can be directed to solve through the linear least-square error method. Additionally, the temperature histories at every node in the direct problems can be obtained. From it the steady time can be determined. It is also for studying the time interval and the inverse values relationship. Furthermore, the accuracy of the estimation of the unknown conditions from the knowledge of the temperature with containing measurement errors are examined at measurement points.

DESCRIPTION OF THE PROPOSED METHOD

Consider an infinite long hollow cylinder, $a \leqslant r \leqslant b$, shown as Fig. 1, with constant thermal properties. This cylinder originally had a zero temperature. At a specific time, an initial condition $g(r, \theta)$ is applied to the cylinder at $t = 0$. One unknown temperature function $f(\theta)$ is applied to the inner surface ($r = a$), and a heat flux $q(\theta)$, at the outer surface ($r = b$). A dimensionless mathematical formulation of the heat conduction problem is presented as:

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 T}{\partial \theta^2}=\frac{\partial T}{\partial \tau} \quad A \leqslant R \leqslant 1, \quad 0 \leqslant \theta \leqslant 2\pi, \quad \tau > 0 \quad (1)$$

$$T(R,\theta,0) = G(r,\theta) \quad A \leqslant R \leqslant 1, \quad 0 \leqslant \theta \leqslant 2\pi, \quad \tau = 0 \quad (2)$$

$$T(A,\theta,\tau) = F(\theta) \quad R = A, \quad 0 \leqslant \theta \leqslant 2\pi, \quad \tau > 0 \quad (3)$$

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NOMENCLATURE

A	matrix, the function of thermal properties
A	dimensionless inner radius $= a/b$
B	the coefficient matrix of θ
$F(\theta)$	dimensionless inner boundary condition
$G(R,\theta)$	dimensionless initial condition
$Q(\theta)$	dimensionless outer boundary condition
R	the reverse matrix of the inverse problem
R	dimensionless radial coordinates
$T(R,\theta,\tau)$	dimensionless temperature.

Greek symbols

θ	matrix, the function of the boundary and initial conditions
$\bar{\theta}$	the coefficient vector of $F(\theta)$, $G(R,\theta)$ and $Q(\theta)$
λ	the probability of a random value
σ	measurement error
τ	dimensionless time.

Subscripts

i	index of dimensionless radial coordinate
j	index of dimensionless angular coordinate
k	index of dimensionless time coordinate.

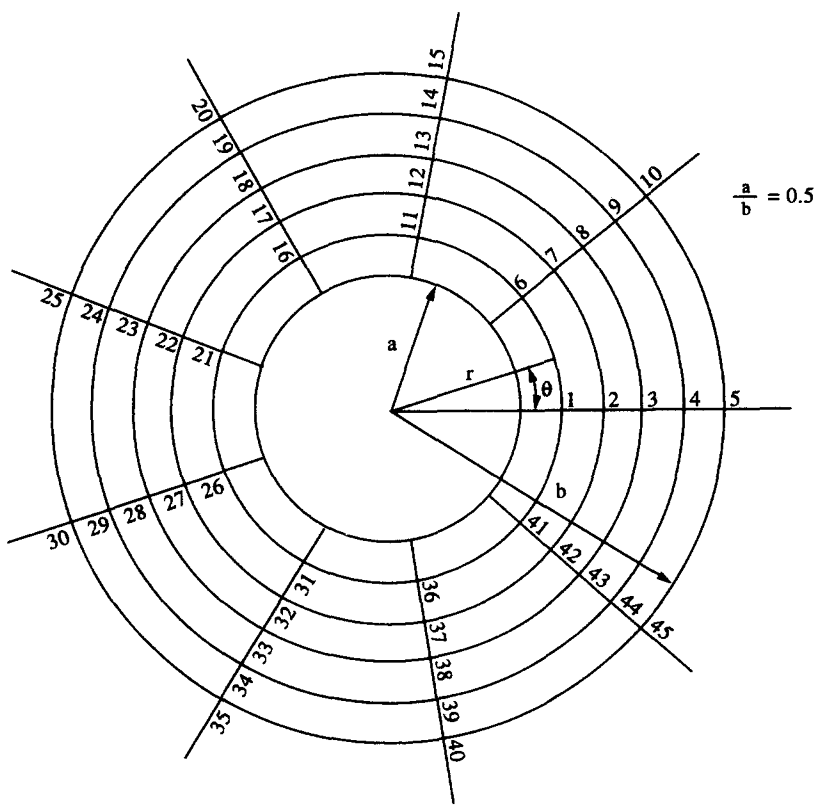

Fig. 1.

$$-\left.\frac{\partial T}{\partial R}\right|_{R=1} = Q(\theta) \quad R=1, \quad 0 \leqslant \theta \leqslant 2\pi, \quad \tau > 0 \tag{4}$$

the various dimensionless parameters in the equations above are defined as follows:

$$A = a/b, \quad R = r/b,$$

$$T = \frac{\bar{T}-\overline{T_\infty}}{q_0 b/k}, \quad G(R,\theta) = \frac{g(r,\theta)-\overline{T_\infty}}{q_0 b/k},$$

$$F(\theta) = \frac{f(\theta)-\overline{T_\infty}}{q_0 b/k}, \quad Q(\theta) = \frac{q(\theta)}{q_0}, \quad \tau = \frac{t\alpha}{b^2}$$

where $\overline{T_\infty}$ is ambient temperature, q_0 is reference heat flux.

This inverse problem is to identify the applied unknown temperature $G(R,\theta)$, $F(\theta)$, and heat flux $Q(\theta)$, from the temperature measurements taken at the interior points of the cylinder.

Suppose that the applied surface temperature $G(R,\theta)$, $F(\theta)$, and heat flux $Q(\theta)$, are represented as the following series forms in the problem domain:

$$G(R,\theta) = \sum_{i=0} a_i \xi_i(R,\theta) \tag{5}$$

$$F(\theta) = \sum_{i=0} b_i \phi_i(\theta) \tag{6}$$

$$Q(\theta) = \sum_{i=0} b_i \zeta(\theta) \tag{7}$$

where $\xi_i(R)$, $\phi_i(\tau)$, and $\zeta_i(\theta)$ can be any non-singular function in the problem domain.

For illustration, the implicated finite-difference method is employed to demonstrate the analysis process. After discretization, the above governing equation combined with the $G(R,\theta)$, $F(\theta)$ and $Q(\theta)$, can be expressed as the following recursive forms:

$$\frac{1}{\Delta R^2}(T_{i-1,j,k} - 2T_{i,j,k} + T_{i+1,j,k}) + \frac{1}{R_i}\frac{1}{2\Delta R}(T_{i+1,j,k} - T_{i-1,j,k}) + \frac{1}{R_i^2}\frac{1}{(\Delta\theta)^2}(T_{i,j-1,k} - 2T_{i,j,k} + T_{i,j+1,k}) = \frac{1}{\Delta\tau}(T_{i,j,k} - T_{i,j,k-1}) \tag{8}$$

where ΔR and $\Delta\theta$ are the increments in the spatial coordinates and $\Delta\tau$ is the increment in the time domain, i is the ith grid along the R coordinate, j is the jth grid along the θ coordinate, k is the kth grid along the time coordinate and $T_{i,j,k}$ is the temperature at the grid point (i,j,k).

Using the recursive forms, a matrix equation can be expressed as

$$\mathbf{AT} = \theta \tag{9}$$

where the $\mathbf{A}$ matrix is the function of thermal properties and the scale of the position and time. The components of $\mathbf{T}$ are the temperature in discretized points, and the components of θ are the function of the boundary conditions, namely the coefficients of $G(R,\theta)$, $F(\theta)$ and $Q(\theta)$.

For the inverse problems, $\mathbf{A}$ can be constructed according to the known physical model and numerical methods and $\mathbf{T}$ can be measured by the thermocouples. The coefficients of $G(R,\theta)$, $F(\theta)$ and $Q(\theta)$ are the main tasks to resolve. Decoupling the coefficients of $G(R,\theta)$, $F(\theta)$ and $Q(\theta)$ from θ will transfer the direct formulation to the following inverse forms:

$$\mathbf{AT} = \mathbf{B}\bar{\theta} \tag{10}$$

where $\theta = \mathbf{B}\bar{\theta}$, $\mathbf{B}$ is the coefficient matrix of $\bar{\theta}$ and $\bar{\theta}$ is the coefficient vector of $G(R,\theta)$ $F(\theta)$ and $Q(\theta)$, then $\bar{\theta}$ can be solved by the linear-squares error method as follows:

$$\bar{\theta} = [(\mathbf{A}^{-1}\mathbf{B})^{\mathrm{T}}(\mathbf{A}^{-1}\mathbf{B})]^{-1}(\mathbf{A}^{-1}\mathbf{B})^{\mathrm{T}}\mathbf{T} \tag{11}$$

$[(\mathbf{A}^{-1}\mathbf{B})^{\mathrm{T}}(\mathbf{A}^{-1}\mathbf{B})]^{-1}(\mathbf{A}^{-1}\mathbf{B})^{\mathrm{T}}$ is the reverse matrix of the inverse problems and denoted as $\mathbf{R}$.

Equation (11) is assumed to measure all discretized points in the problems. The realistic experimental approach is to measure only the few points or one position point in the problem. We can construct the part of matrices $\mathbf{R}$, $\mathbf{T}$ and $\bar{\theta}$ corresponding to the measuring positions and times in order to estimate the unknown conditions of the problem.

According to the above derivation, it is possible to identify whether the solution is unique or not. The method by which to identify the properties of the solution is based on the theory of linear algebra, which will be shown in the following descriptions. If the rank of reverse matrix is less than the number of elements of the coefficient vector, the number of measurements in time domains need to be increased. Furthermore, if the rank of reverse matrix is equal to the number of elements of the coefficient vector, the perpendicular distance from $\bar{\theta}$ to the column space of $\mathbf{A}^{-1}\mathbf{B}$ need to be checked. If the distance is vanished, the solution becomes unique.

RESULTS AND DISCUSSION

The problem contained a number of examples to verify the accuracy, efficiency, and versatility of the proposal method for simultaneously estimating the initial and boundary conditions. The direct problem, the special interval $0.5 \leqslant R \leqslant 1.0$ is divided into five intervals, and $0 \leqslant \theta \leqslant 2\pi$ is divided into 10 intervals. The iteration step corresponds to a mesh size of $\Delta R = 0.1$ and $\Delta\theta = 2\pi/9$. Equations (8)–(11) were applied to obtain the temperature histories at the nodes of the hollow pipe. The temperature histories is assumed to measure all discretized points in the inverse problem. The simulated temperature in all examples is presumed to contain measurement errors. In other words, the random errors of measurements are added to the exact temperature computed from the solution of direct problem. Thus, it can be written as

$$\theta_{\text{measurement}} = \theta_{\text{exact}} + \lambda\sigma. \tag{12}$$

For normal distribution errors, the probability of a random value, λ, lying in the range $-3.0 \leqslant \lambda \leqslant 3.0$ is 99.43%.

The accuracy of the estimation of the unknown conditions from the knowledge of the temperature at measurement points are examined. As a result, the estimated solutions without containing measurement error ($\sigma = 0$) converged to the solutions solved by finite-difference method for all examples. Furthermore, the solutions are unique through the proposed verifying method. Detailed descriptions for the problem are shown as follows

Example 1

The unknown initial condition $[g(r,\theta)]$ and the boundary term $[f(\theta)$ and $q(\theta)]$ are expressed in the following form:

$$g(r,\theta) = 0.0 \quad \text{at } t = 0$$

$$f(\theta) = \overline{T_\alpha} + \overline{T_0}\sin\theta \quad \text{at } r = a$$

$$q(\theta) = -q_0\sin\theta = -K\frac{\partial \overline{T}}{\partial r} \quad \text{at } r = b$$

the corresponding dimensionless form can be described as:

$$G(R,\theta) = 0 \quad \text{at } \tau = 0$$

$$F(\theta) = \frac{\overline{T_0}}{q_0 b/K}\sin\theta \quad \text{at } R = A$$

$$Q(\theta) = \frac{q}{q_0} = -\sin\theta = -\frac{\partial T}{\partial R} \quad \text{at } R = 1$$

$$let\ \overline{T_0}/(q_0 b/K) = 0.8.$$

The temperature profile in direct problem for various time can be obtained by using equations (8)–(11). The results are shown in Fig. 2, it approaches steady state for $\tau \geqslant 1.2$.

For this example, the steady-state has been solved by Tseng *et al.* [20]. Figure 3 shows a comparison of the present estimate with those given by Tseng *et al.* The agreements are quite excellent. However, the 2 nodes temperature histories (20 time-step) are used in the present study, it can estimate the initial and boundary conditions, while in Tseng' study [20], a total of 80 interior measurements are needed to estimate the boundary conditions only. This implies that the present method is more powerful than Tseng's DSC method and hence, the proposed method is much more effective for inverse heat coeducation problems. As illustrated in Table 1, for the case of reflected measuring errors, very good approximation can be attained, even for two measurement points only.

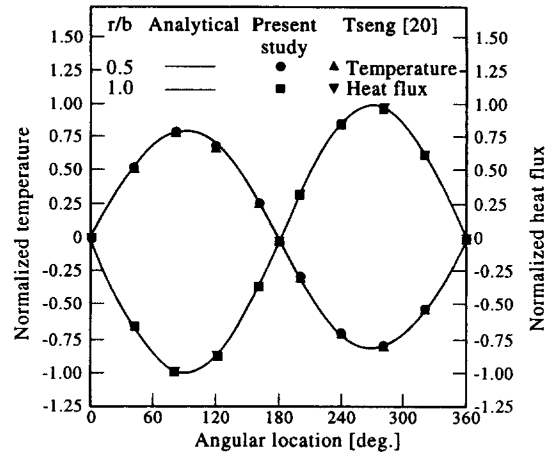

Fig. 3.

Figure 4 shows a comparison of surface temperature and heat flux for $\sigma = 0.0$, 1.0 and 5.0% with 18 measurement points in example 1. The input data without measurements errors, is shown as a solid line. Maximum discrepancies in temperature at the inner surface are 1.13 and 7.91% for the 1.0 and 5.0 error cases, respectively, the corresponding discrepancies in the heat flux are even greater, at 2.77 and 13.84% for the 1.0 and 5.0% error cases, respectively. The discrepancy magnitudes in both temperature and heat flux are directly proportional to the possible measuring errors evolved. The present results confirm that the inverse values being extremely sensitive to measuring error is one of the inherent characteristics of IHCP, as mentioned by Beek *et al.* [20] and Hense [21].

The estimated results with errors 0.0, 1.0 and 5.0% at different time intervals are shown in Tables 2–4, respectively. When measurement error is free, the estimated values are very close to the exact solution, and independent of time coordinate. At $\sigma = 1.0\%$, the

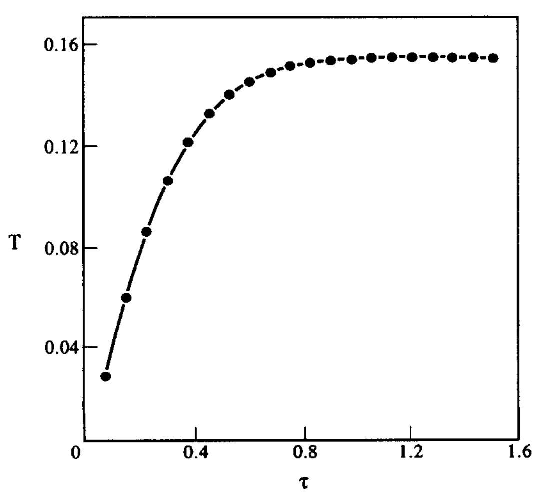

Fig. 2.

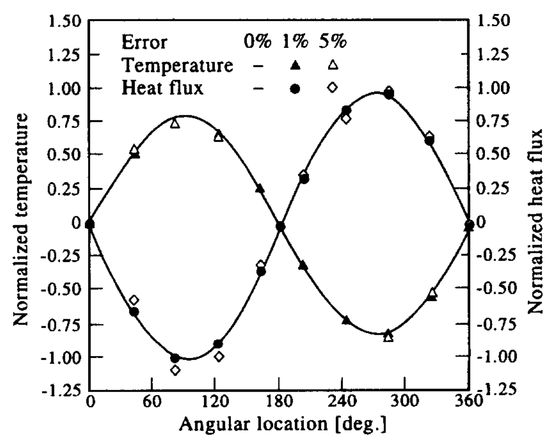

Fig. 4.

Table 1. Estimate initial condition $[G(R, \theta)]$ and boundary condition $[F(\theta)$, and $Q(\theta)]$ for example 1 without measured error, time from 0.075 to 1.5, and $R = 0.8$

	$F(\theta)$				$Q(\theta)$			
		Estimate				Estimate		
θ (deg)	Exact	2 points	3 points	9 points	Exact	2 points	3 points	9 points
0	0.0	-8.134×10^{-9}	-60244×10^{-8}	-1.482×10^{-8}	0.0	-3.362×10^{-8}	-1.167×10^{-8}	4.0432×10^{-8}
40	0.5142300	0.5142297	0.5142306	0.5142297	−0.642787	−0.642787	−0.642785	−0.642787
80	0.7878462	0.7878453	0.7878465	0.7878460	−0.984807	−0.984809	−0.984806	−0.984807
120	0.6928203	0.6928211	0.6928209	0.6928209	−0.866025	−0.866025	−0.866026	−0.866026
160	0.2736161	0.2736170	0.2736187	0.2736177	−0.342020	−0.342024	−0.342020	−0.342022
200	−0.273616	−0.273614	−0.273614	−0.273614	0.3420201	0.3420174	0.3420173	0.3420177
240	−0.692820	−0.692817	−0.692818	−0.692819	0.8660254	0.8660266	0.8660254	0.8660237
280	−0.787846	−0.787845	−0.787843	−0.787846	0.9848077	0.9848100	0.9848141	0.9848084
320	−0.514230	−0.514233	−0.514232	−0.514232	0.6427876	0.6427899	0.6427906	0.6427909
Max. error	0.0	3.2×10^{-5}	1.36×10^{-6}	1.6×10^{-7}	0.0	3.3×10^{-6}	3.1×10^{-6}	1.9×10^{-7}
$G(R,\theta)$	0.0	1.723×10^{-10}	1.6754×10^{-9}	8.266×10^{-10}	0.0	1.723×10^{-10}	1.675×10^{-9}	8.266×10^{-10}

Table 2. Estimate initial condition $[G(R, \theta)]$ and boundary condition $[F(\theta)$, and $Q(\theta)]$, for $\tau = 0.075 \sim 1.5$, $\tau = 0.075 \sim 0.75$, and $\tau = 0.75 \sim 1.5$, in example 1 without measured error, 3 measuring points

	$F(\theta)$				$Q(\theta)$			
		Estimate				Estimate		
θ (deg)	Exact	$\tau=0.075\sim1.5$	$\tau=0.075\sim0.7$	$\tau=0.075\sim1.5$	Exact	$\tau=0.075\sim1.5$	$\tau=0.075\sim0.7$	$\tau=0.075\sim1.5$
0	0.0	-60244×10^{-8}	-5.242×10^{-8}	-4.726×10^{-8}	0.0	-1.167×10^{-8}	1.6808×10^{-8}	1.1535×10^{-8}
40	0.5142300	0.5142306	0.5142306	0.5142305	−0.642787	−0.642785	−0.642785	−0.642785
80	0.7878462	0.7878465	0.7878465	0.7878475	−0.984805	−0.984806	−0.984806	−0.984804
120	0.6928203	0.6928209	0.6928209	0.6928209	−0.866025	−0.866026	−0.866026	−0.866026
160	0.2736161	0.2736187	0.2736183	0.2736173	−0.342020	−0.342020	−0.342021	−0.342023
200	−0.273616	−0.273614	−0.273614	−0.273614	0.3420201	0.3420178	0.3420178	0.3420176
240	−0.692820	−0.692818	−0.692818	−0.692817	0.8660254	0.8660254	0.8660250	0.8660262
280	−0.787846	−0.787843	−0.787844	−0.787846	0.9848077	0.9848141	0.9848126	0.9848086
320	−0.514230	−0.514232	−0.514233	−0.513230	0.6427876	0.6427906	0.6427874	0.6427953
Max. error	0.0	3.2×10^{-5}	2.6×10^{-6}	1.36×10^{-6}	0.0	3.3×10^{-6}	6.4×10^{-7}	3.1×10^{-6}
$G(R,\theta)$	0.0	1.6754×10^{-9}	1.286×10^{-9}	1.2563×10^{-9}	0.0	1.6754×10^{-9}	1.2866×10^{-9}	1.2563×10^{-9}

Table 3. Estimate initial condition $[G(R, \theta)]$ and boundary condition $[F(\theta)$, and $Q(\theta)]$, for $\tau = 0.075 \sim 1.5$, $\tau = 0.075 \sim 0.75$, and $\tau = 0.75 \sim 1.5$, in example 1, with 1% measured error, 18 measuring points

	$F(\theta)$				$Q(\theta)$			
		Estimate				Estimate		
θ (deg)	Exact	$\tau=0.075\sim1.5$	$\tau=0.07\sim0.7$	$\tau=0.075\sim1.5$	Exact	$\tau=0.075\sim1.5$	$\tau=0.075\sim0.7$	$\tau=0.075\sim1.5$
0	0.0	1.295×10^{-3}	8.8391×10^{-4}	1.1327×10^{-3}	0.0	3.8852×10^{-4}	-4.048×10^{-3}	-4.365×10^{-3}
40	0.5142300	0.5224885	0.5211589	0.5189049	−0.642787	−0.625438	−0.630550	−0.631787
80	0.7878462	0.7789201	0.7919694	0.7879205	−0.984807	−1.003938	−0.982435	−0.992486
120	0.6928203	0.69881345	0.6801135	0.6874025	−0.866025	−0.890007	−0.897082	−0.905507
160	0.2736161	0.2741542	0.2818738	0.2804559	−0.342020	−0.332023	−0.317750	−0.320948
200	−0.273616	−0.271711	−0.272478	−0.271666	0.3420201	0.3486107	0.3555941	0.3525880
240	−0.692820	−0.692407	−0.682803	−0.687636	0.8660254	0.8520062	0.8623446	0.8508267
280	−0.787846	−0.795160	−0.791878	−0.790539	0.9848077	0.9889499	0.9785556	0.9918379
320	−0.514230	−0.506093	−0.496116	−0.503388	0.6427876	0.6517354	0.6553022	0.6404372
Max. error	0.0	0.0089	0.0181136	0.010842	0.0	0.0239822	0.0310570	0.0394816
		(1.13%)	(3.52%)	(2.11%)		(3.57%)	(3.57%)	(4.56%)
$G(R,\theta)$	0.0	2.5737×10^{-3}	-2.007×10^{-4}	-1.908×10^{-4}	0.0	2.5737×10^{-3}	-2.007×10^{-4}	-1.908×10^{-4}

Table 4. Estimate initial condition $[G(R, \theta)]$ and boundary condition $[F(\theta)$, and $Q(\theta)]$, for $\tau = 0.075 \sim 1.5$, $\tau = 0.075 \sim 0.75$, and $\tau = 0.75 \sim 1.5$, in example 1, with 5% measured error, 18 measuring points

	$F(\theta)$				$Q(\theta)$			
		Estimate				Estimate		
θ (deg)	Exact	$\tau = 0.075 \sim 1.5$	$\tau = 0.075 \sim 0.75$	$\tau = 0.075 \sim 1.5$	Exact	$\tau = 0.075 \sim 1.5$	$\tau = 0.075 \sim 0.75$	$\tau = 0.075 \sim 1.5$
0	0.0	6.4790×10^{-3}	4.4195×10^{-3}	50663×10^{-3}	0.0	1.9425×10^{-3}	-2.024×10^{-2}	-2.162×10^{-2}
40	0.5142300	0.5555238	0.5488757	0.5376057	−0.642787	−0.556043	−0.581605	−0.587786
80	0.7878462	0.7432166	0.8084630	0.7882186	−0.984807	−1.080461	−0.972949	−1.023200
120	0.6928203	0.6693886	0.6292837	0.6207289	−0.866025	−0.985933	−1.021307	−1.063430
160	0.2736161	0.2762998	0.3148981	0.3078084	−0.342020	−0.292030	−0.220663	−0.236656
200	−0.273616	−0.264099	−0.267936	−0.263873	0.3420201	0.3749827	0.4098997	0.3948694
240	−0.692820	−0.690760	−0.642774	−0.666904	0.8660254	0.7959363	0.8476284	0.7900388
280	−0.787846	−0.824416	−0.808004	−0.801309	0.9848077	1.0055160	0.9535447	1.0199562
320	−0.514230	−0.473534	−0.423651	−0.460013	0.6427876	0.6875131	0.7053470	0.6310223
Max. error	0.0	0.0406956	0.0905787	0.072091	0.0	0.1199081	0.1552820	0.1974050
		(7.91%)	(17.61%)	(10.4%)		(13.84%)	(17.93%)	(22.79%)
$G(R, \theta)$	0.0	1.2868×10^{-2}	-1.003×10^{-3}	-9.541×10^{-4}	0.0	1.286×10^{-2}	-1.003×10^{-3}	-9.541×10^{-4}

maximum discrepancies in temperature are 1.132, 3.52 and 2.11%, the maximum in heat flux are 2.77, 3.57 and 4.56%, for the time interval, $\Delta\tau = 0.075 \sim 1.5$, $0.075 \sim 0.75$ and $0.75 \sim 1.5$, respectively. When $\sigma = 5.0\%$, the maximum deviations in temperature are 7.91, 17.61 and 10.4%, the maximum discrepancies in heat flux are 13.84, 17.93 and 22.79%, for the time interval, $\Delta\tau = 0.075 \sim 1.5$, $0.075 \sim 0.75$ and $0.75 \sim 1.5$, respectively.

Example 2

The unknown boundary term $[F(\theta)$, and $Q(\theta)]$ is the same as in Example 1, but the initial condition $[G(R, \theta)$ at $\tau = 0]$ has no special form, it is expressed as follows:

$$G(0.5, \theta) = 0.0$$

$$G(0.6, \theta) = 0.58778525$$

$$G(0.7, \theta) = 0.95105651$$

$$G(0.8, \theta) = 0.95105651$$

$$G(0.9, \theta) = 0.58778525$$

$$G(1.0, \theta) = 0.0.$$

The temperature profile in direct problem are shown in Fig. 5, the state is steady for $\tau \geqslant 1.2$. Without considering the measurement errors, the estimate values, as shown in Table 5, have very good approximation even for two measuring points only.

Figure 6 shows a comparison of surface temperature, and heat flux for $\sigma = 0.0$, 1.0 and 5.0% with 18 measuring points. The input data without measurement errors, is shown as a solid line. Maximum discrepancies in temperature at the inner surface are 3.15 and 6.22% for the 1.0 and 5.0% error cases, respectively, while the maximum discrepancies in heat flux are 8.7 and 15.34% for the 1.0 and 5.0% error cases, respectively. Table 6 shows the corresponding maximum discrepancies in initial conditions are even greater, at 0.0002, 0.96 and 34.99% for the 0.0, 1.0

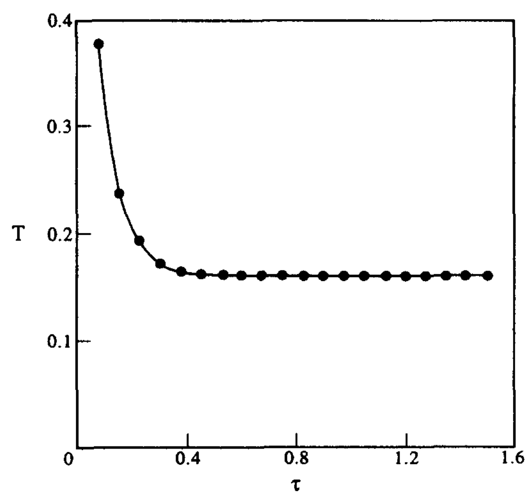

Fig. 5.

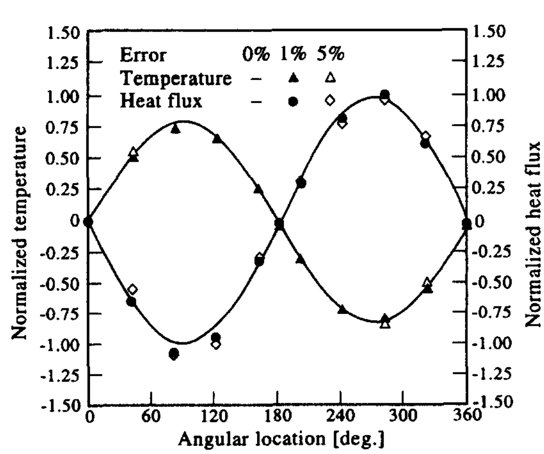

Fig. 6.

Table 5. Estimate initial condition $[G(R, \theta)]$ and boundary condition $[F(\theta)$, and $Q(\theta)]$ for example 2 without measured error, time from 0.075 to 1.5, and $R = 0.8$

	$F(\theta)$				$Q(\theta)$			
		Estimate				Estimate		
θ (deg)	Exact	2 points	3 points	9 points	Exact	2 points	3 points	9 points
0	0.0	8.731×10^{-6}	1.0210×10^{-3}	3.307×10^{-8}	0.0	1.8127×10^{-5}	-2.5152×10^{-4}	-6.865×10^{-8}
40	0.5142300	0.5143150	0.5153843	0.5139164	−0.642787	−0.642613	−0.640445	−0.643423
80	0.7878462	0.7879255	0.7890025	0.7875328	−0.984807	−0.984646	−0.982463	−0.985443
120	0.6928203	0.6929107	0.6939631	0.6925077	−0.866025	−0.865841	−0.863703	−0.866662
160	0.2736161	0.2736951	0.2747834	0.2733045	−0.342020	−0.341867	−0.339664	−0.342658
200	−0.273616	−0.273530	−0.272472	−0.273927	0.3420201	0.3421876	0.3443413	0.3413817
240	−0.692820	−0.692729	−0.691656	−0.693132	0.8660254	0.8662092	0.8683742	0.8653873
280	−0.787846	−0.787768	−0.786698	−0.788159	0.9848077	0.9849645	0.9871478	0.9841725
320	−0.514230	−0.514144	−0.513080	−0.514546	0.6427876	0.6429725	0.6451182	0.6421548
Max. error	0.0	8.561×10^{-5}	1.1494×10^{-3}	3.1367×10^{-4}	0.0	1.567×10^{-4}	2.322×10^{-3}	6.384×10^{-4}
		(0.02%)	(0.22%)	(0.06%)		(0.02%)	(0.27%)	(0.19%)

	$G(R, \theta)$			
		Estimate		
R	Exact	2 points	3 points	9 points
0.6	0.58778525	0.58775861	0.58881026	0.58758933
0.7	0.95105651	0.95115641	0.95107261	0.95099708
0.8	0.95105651	0.95105575	0.95106280	0.95105646
0.9	0.58778525	0.58770673	0.58772861	0.58786319
1.0	0.0	−0.00000456	−0.00067381	0.00015165
Max. error	0.0	7.852×10^{-5} (0.13%)	1.025×10^{-3} (0.17%)	1.593×10^{-4} (0.16%)

Table 6. Comparison initial condition $[G(R, \theta)]$, for $\sigma = 0\%$, $\sigma = 1\%$, and $\sigma = 5\%$, $\tau = 0.075 \sim 1.5$, in example 2, with 18 measuring points

	$G(R, \theta)$			
		Estimate		
R	Exact	$\sigma = 0\%$	$\sigma = 1\%$	$\sigma = 5\%$
0.6	0.58778525	0.58778499	0.58214325	0.45245674
0.7	0.95105651	0.95105631	0.957761801	1.28386210
0.8	0.95105651	0.95105681	0.95239247	0.64226435
0.9	0.58778525	0.58778636	0.58324153	0.66506224
1.0	0.0	0.00000018	−0.0041205	0.20603362
Max. error	0.0	1.11×10^{-6} (0.0002%)	6.56×10^{-3} (0.96%)	3.328×10^{-1} (34.99%)

and 5.0% error cases, respectively. Table 7 also shows a comparison of surface temperature, and heat flux for $\sigma = 0.0$, 1.0 and 5.0% with 45 measuring points. Maximum discrepancies in temperature at the inner surface are 0.0006, 0.71 and 3.49% for the 0.0, 1.0 and 5.0% error cases, respectively, the maximum discrepancies in heat flux are 0.007, 0.83 and 4.15% for the 0.0, 1.0 and 5.0% error cases, respectively. The corresponding maximum discrepancies in initial conditions are 0.0002, 0.69 and 18.90% for the 0.0, 1.0 and 5.0% error cases, respectively. The discrepancy magnitudes in the temperature, heat flux, and initial condition are directly proportional to the size of measurement error, the inverse values are sensitive to measurement error. The present results confirm that the inverse values being extremely sensitive to measurement error is one of the inherent characteristics of IHCP. By increasing measurement points, the accuracy of the estimate value increases.

CONCLUSION

The proposed method has been introduced for solving a 2-D hollow cylinder inverse conduction problem. A direct inverse formulation is constructed using the reverse matrix which derives from the governing equa-

Table 7. Estimate initial condition $[G(R, \theta)]$ and boundary condition $[F(\theta)$, and $Q(\theta)]$, for $\sigma = 0\%$, $\sigma = 1\%$, and $\sigma = 5\%$, $\tau = 0.075 \sim 1.5$, in example 2, with 45 measuring points

	$F(\theta)$				$Q(\theta)$			
		Estimate				Estimate		
θ (deg)	Exact	$\sigma = 0\%$	$\sigma = 1\%$	$\sigma = 5\%$	Exact	$\sigma = 0\%$	$\sigma = 1\%$	$\sigma = 5\%$
0	0.0	1.004×10^{-9}	1.4880×10^{-4}	7.4400×10^{-4}	0.0	7.0518×10^{-9}	-2.467×10^{-3}	-1.233×10^{-2}
40	0.5142300	0.5142297	0.5131821	0.5089919	−0.642787	−0.642787	−0.640229	−0.629996
80	0.7878462	0.7878460	0.7862000	0.7796159	−0.984807	−0.984807	−0.992995	−1.025745
120	0.6928203	0.6928209	0.6879825	0.6686285	−0.866025	−0.866026	−0.874713	−0.909464
160	0.2736161	0.2736178	0.2733922	0.2724897	−0.342020	−0.342022	−0.341835	−0.341090
200	−0.273616	−0.273614	−0.275240	−0.281744	0.3420201	0.3420177	0.3392869	0.3283636
240	−0.692820	−0.692819	−0.689546	−0.676455	0.8660254	0.8660237	0.8713882	0.8928462
280	−0.787846	−0.787746	−0.789339	−0.795312	0.9848077	0.9848084	0.9859052	0.9902927
320	−0.514230	−0.514232	−0.512341	−0.504776	0.6427876	0.6427910	0.6477436	0.6675539
Max. error	0.0	2.736×10^{-6}	4.9378×10^{-3}	2.4191×10^{-2}	0.0	2.42×10^{-5}	8.1847×10^{-3}	4.0938×10^{-2}
		(0.0006%)	(0.71%)	(3.49%)		(0.007%)	(0.83%)	(4.15%)

	$G(R, \theta)$			
		Estimate		
R	Exact	$\sigma = 0\%$	$\sigma = 1\%$	$\sigma = 5\%$
0.6	0.58778525	0.58778499	0.59244097	0.70902689
0.7	0.95105651	0.95105631	0.95033039	0.94742670
0.8	0.95105651	0.95105681	0.95878477	0.98969660
0.9	0.58778525	0.58778636	0.56498039	0.47375651
1.0	0.0	0.00000018	0.00120777	0.06038165
Max. error	0.0	1.11×10^{-6} (0.0002%)	4.0557×10^{-3} (0.69%)	1.1140×10^{-1} (18.9%)

tion, initial and boundary conditions. Two examples have been used to show the robustness of the proposed method. From the results, it appears that the proposed method without measurement error the exact solution can be found when only few points (two, or three) are measured, and the estimated values are stability (transient or steady-state). When the measurement errors are included, it is suggested that more measurement points (18, or 45) are to be adopted for a better result in the problem, and it can also be found that the present method gives a little effect of the measurement time interval (transient or steady-state) on the estimates. This implies that the present model offers a great deal of flexibility. After all, the results confirm that the proposed method is effective for inverse heat conduction problems.

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